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LETTER TO THE EDITOR

Knitting ansatz and solutions to Yang–Baxter equation*

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Abstract. We suggest a new method, named the *knitting ansatz*, to generate solutions to the Yang–Baxter equation with lower-dimensional representations of the braid group. To support our ansatz, we work out an example of a new 16×16 R-matrix constructed according to this idea, with two 4×4 braid group representations of familiar 6-vertex type with different q-parameters.

1. Introduction

A braid is one of the simplest kinds of knitting craft. In this letter, we will work on the *knitting* of braid group representations which gives solutions of the Yang–Baxter equation [1, 2]. Acurately, we suggest an ansatz to construct a solution of the Yang–Baxter equation from two (or more) lower-dimensional braid group representations, i.e. the *knitting ansatz*:

> If S_1 and S_2 are both braid group representations, then $S = S_1 \otimes S_2$ gives a solution of the Yang–Baxter equation through a proper Yang–Baxterization approach (of which a brief review is given in the text).

This *knitting* construction is an ansatz because the Yang–Baxterization approach we apply is an ansatz (see the later section for a discussion). It can easily be seen that the knitting procedure can be applied to more than two braid group representations, S_i with $(i = 1, 2, \ldots, N)$.

There are many schools of Yang–Baxterization approach [3–5], and the approach we apply in this letter is a modified version [7] of the theory given by Wang, Ge and Xue [6, 5].

A proper parametrization of S will give a solution of the Yang–Baxter equation. But generically, such solutions are trivial, in that they are simply tensor products of the solutions of the Yang–Baxter equation that are parametrized separately. Later in this letter, we will work out an example that gives a non-trivial solution. Actually, this non-trivial solution is a new solution to the Yang–Baxter equation.

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2. A generalized approach of Yang–Baxterization

This section is devoted to a brief review of a Yang–Baxterization approach, developed by Ge *et al* [5, 6], and generalized by one of the present authors in [7]. This approach provides a method (more accurately, an ansatz), for the parametrization of a braid group representation into a trigonometric solution of the Yang–Baxter equation.

The Yang–Baxter equation takes the form

$$\hat{R}_{12}(x)\hat{R}_{23}(xy)\hat{R}_{12}(y) = \hat{R}_{23}(y)\hat{R}_{12}(xy)\hat{R}_{23}(x) \tag{1}$$

where $\hat{R}_{12}(x) = \hat{R}(x) \otimes \mathbb{1}$ and $\hat{R}_{23}(x) = \mathbb{1} \otimes \hat{R}(x)$. Let S be a braid group representation with m distinct eigenvalues λ_i with $i = 1, 2, \ldots, m$ and let it obey a character equation

$$\prod_{i=1}^{m}(S - \lambda_i) = 0 \tag{2}$$

then the braid group representation S is expanded via the projectors

$$P_i = \prod_{j\neq i}\frac{(S-\lambda_j)}{(\lambda_i - \lambda_j)} \tag{3}$$

i.e.

$$S = \sum_{i=1}^{m}\lambda_i P_i\,. \tag{4}$$

The trigonometric Yang–Baxterization gives the solutions of the Yang–Baxter equation with a spectral decomposition of the following form:

$$\hat{R}(x) = \sum_{i=1}^{m}\Lambda_i(x)P_i \tag{5}$$

where $\Lambda_i(x)$ are m functions to be determined, upon the requirement called the standard initial condition:

$$\hat{R}(1) \propto \mathbb{1}\,. \tag{6}$$

Generally

$$\Lambda_i(x) = \prod_{j=1}^{i-1}\left(1 + x\frac{\tilde{\lambda}_j}{\tilde{\lambda}_{j+1}}\right)\prod_{j=i}^{m-1}\left(x + \frac{\tilde{\lambda}_j}{\tilde{\lambda}_{j+1}}\right). \tag{7}$$

Note that $\tilde{\lambda}_i$ with $i = 1, 2, \ldots, m$ take distinct values in the set $\{\lambda_1, \lambda_1, \ldots, \lambda_m\}$. In fact this approach is an ansatz because it only guarantees that $\hat{R}(x)$ in (5) satisfies the initial condition (6), and a proper alignment of the eigenvalues is needed for $\hat{R}(x)$ to satisfy the Yang–Baxter equation.

It is easy to see that when $x = 1$

$$\Lambda_1(1) = \Lambda_2(1) = \cdots = \Lambda_m(1) = \prod_{j=1}^{m-1}\left(1 + \frac{\tilde{\lambda}_j}{\tilde{\lambda}_{j+1}}\right) \tag{8}$$

or

$$\hat{R}(1) = \Lambda_1(1)\cdot\mathbb{1}\,. \tag{9}$$

We would like to stress that the above ansatz is not unique. Actually, it has been pointed out in [7] that, under the initial condition, equation (5) can be generalized as

$$\Lambda_i(x) = \prod_{j=1}^{i-1}\left(1 + x^{m_j}\frac{\tilde{\lambda}_j}{\tilde{\lambda}_{j+1}}\right)\prod_{j=i}^{m-1}\left(x^{m_j} + \frac{\tilde{\lambda}_j}{\tilde{\lambda}_{j+1}}\right) \qquad m_j \in \mathbb{Z}\,. \tag{10}$$

A special case with three eigenvalues was discussed in [7]; here we are dealing with an interesting case of four eigenvalues. For convenience in later discussions we give the explicit form of (10) in this four-eigenvalues case:

$$\begin{aligned}
\Lambda_1(x) &= (\lambda_1/\lambda_2 + x)(\lambda_2/\lambda_3 + x^2)(\lambda_3/\lambda_4 + x) \\
\Lambda_2(x) &= (\lambda_1 x/\lambda_2 + 1)(\lambda_2/\lambda_3 + x^2)(\lambda_3/\lambda_4 + x) \\
\Lambda_3(x) &= (\lambda_1 x/\lambda_2 + 1)(\lambda_2 x^2/\lambda_3 + 1)(\lambda_3/\lambda_4 + x) \\
\Lambda_4(x) &= (\lambda_1 x/\lambda_2 + 1)(\lambda_2 x^2/\lambda_3 + 1)(\lambda_3 x/\lambda_4 + 1) \\
\hat{R}(x) &= \Lambda_1(x) P_1 + \Lambda_2(x) P_2 + \Lambda_3(x) P_3 + \Lambda_4(x) P_4.
\end{aligned} \tag{11}$$

3. The knitting ansatz

Let $\hat{R}_1$ and $\hat{R}_2$ be the corresponding R-matrices Yang–Baxterized from the braid group representations S_1 and S_2, and let S be the direct product of S_1 and S_2. Obviously S is also a representation of a braid group. Yang–Baxterizing S, we can get at least one solution which is the direct product of $\hat{R}_1$ and $\hat{R}_2$. $\hat{R}$ of this kind is not meaningful to any solution constructor and hence is called trivial. We will give a non-trivial example at the end of this section.

3.1. Expressions for S_1 and S_2

Considering two representations S_1 and S_2 of the braid group with different parameters such that

$$S_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q_1 & 0 \\ 0 & q_1 & 1 - q_1^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q_2 & 0 \\ 0 & q_2 & 1 - q_2^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with eigenvalues $1,\ -q_1^2$ and $1,\ -q_2^2$, respectively, the tensor product S of S_1 and S_2 reads

$$S = \begin{bmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & q_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q_{12} & \cdot & \cdot & \cdot \\
\cdot & q_2 & \cdot & \cdot & q_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q_{12} & \cdot & \cdot & q_{13} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q_1 & \cdot & \cdot \\
\cdot & \cdot & q_1 & \cdot & \cdot & \cdot & \cdot & \cdot & q_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & q_{24} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q_2 & \cdot \\
\cdot & \cdot & \cdot & q_{12} & \cdot & \cdot & q_{13} & \cdot & \cdot & q_{24} & \cdot & \cdot & q_{34} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q_1 & \cdot & \cdot & \cdot & \cdot & \cdot & q_4 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q_2 & \cdot & \cdot & q_3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{bmatrix} \tag{12}$$

where $q_3 = 1 - q_2^2,\ q_4 = 1 - q_1^2,\ q_{12} = q_1q_2,\ q_{13} = q_1q_3,\ q_{24} = q_2q_4$ and $q_{34} = q_3q_4$. Obviously the set of eigenvalues of S $1,\ -q_1^2,\ -q_2^2$ and $q_1^2q_2^2$ is the same as the product set of that of S_1 and S_2.

3.2. $\hat{R}_1$ *and* $\hat{R}_2$

Baxterizing S_1 and S_2, we get two R-matrices $\hat{R}_1(x)$ and $\hat{R}_2(x)$ taking the following forms:

$$\hat{R}_1(x) = \begin{bmatrix} xq_1^2-1 & 0 & 0 & 0 \\ 0 & x(q_1^2-1) & q_1(x-1) & 0 \\ 0 & q_1(x-1) & q_1^2-1 & 0 \\ 0 & 0 & 0 & q_1^2x-1 \end{bmatrix}$$

and

$$\hat{R}_2(x) = \begin{bmatrix} x^2q_2^2-1 & 0 & 0 & 0 \\ 0 & x^2(q_2^2-1) & q_2(x^2-1) & 0 \\ 0 & q_2(x^2-1) & q_2^2-1 & 0 \\ 0 & 0 & 0 & q_2^2x^2-1 \end{bmatrix}$$

with eigenvalues $q_1^2 - x,\ xq_1^2 - 1$ and $q_2^2 - x^2,\ x^2q_1^2 - 1$, respectively.

In fact $\hat{R}_2$ can take the same form as $\hat{R}_1$. We select this form for $\hat{R}_2(x)$ only for later use.

The direct product $\hat{R}_\otimes(x)$ of $\hat{R}_1(x)$ and $\hat{R}_2(x)$ is symmetric and the non-zero elements are

$$\begin{aligned}
&\hat{R}_{\otimes 1\ 1} = \hat{R}_{\otimes 6\ 6} = \hat{R}_{\otimes 11\ 11} = \hat{R}_{\otimes 16\ 16} = (x^2q_2^2-1)(xq_1^2-1)\\
&\hat{R}_{\otimes 2\ 2} = \hat{R}_{\otimes 12\ 12} = (xq_1^2-1)(q_2^2-1)x^2\\
&\hat{R}_{\otimes 2\ 5} = \hat{R}_{\otimes 12\ 15} = (xq_1^2-1)(x^2-1)q_2\\
&\hat{R}_{\otimes 3\ 3} = \hat{R}_{\otimes 8\ 8} = (x^2q_2^2-1)(q_1^2-1)x\\
&\hat{R}_{\otimes 3\ 9} = \hat{R}_{\otimes 8\ 14} = (x^2q_2^2-1)(x-1)q_1\\
&\hat{R}_{\otimes 4\ 4} = (q_2^2-1)(q_1^2-1)x^3\\
&\hat{R}_{\otimes 4\ 7} = (x^2-1)(q_1^2-1)xq_2\\
&\hat{R}_{\otimes 4\ 10} = (x-1)(q_2^2-1)x^2q_1\\
&\hat{R}_{\otimes 4\ 13} = \hat{R}_{\otimes 7\ 10} = (x+1)(x-1)^2q_2q_1\\
&\hat{R}_{\otimes 5\ 5} = \hat{R}_{\otimes 15\ 15} = (xq_1^2-1)(q_2^2-1)\\
&\hat{R}_{\otimes 7\ 7} = (q_2^2-1)(q_1^2-1)x\\
&\hat{R}_{\otimes 7\ 13} = (x-1)(q_2^2-1)q_1\\
&\hat{R}_{\otimes 9\ 9} = \hat{R}_{\otimes 14\ 14} = (x^2q_2^2-1)(q_1^2-1)\\
&\hat{R}_{\otimes 10\ 10} = (q_2^2-1)(q_1^2-1)x^2\\
&\hat{R}_{\otimes 10\ 13} = (x^2-1)(q_1^2-1)q_2\\
&\hat{R}_{\otimes 13\ 13} = (q_2^2-1)(q_1^2-1)
\end{aligned}$$

and the set of its eigenvalues is

$$\left\{(xq_1^2-1)(x^2q_2^2-1), -(x^2q_2^2-1)(x-q_1^2), (x-q_1^2)(x^2-q_2^2), -(xq_1^2-1)(x^2-q_2^2)\right\} \tag{13}$$

which is the product set of that of $\hat{R}_1$ and $\hat{R}_2$. This procedure is depicted in figure 1.

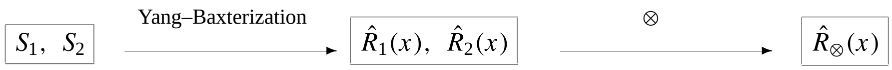

Figure 1. Yang–Baxterization and then tensor production.

3.3. A new solution

As an example, we now Yang–Baxterize S using (11). In general, there are 24 possible cases to get the solutions of the Yang–Baxter equation. If we denote the permutation of these four eigenvalues $1, -q_1^2, -q_2^2, q_1^2q_2^2$ as 1234, then for the following eight cases with permutaions 1243, 1423, 2134, 2314, 3241, 3421, 4132 and 4312 we get the solutions of the Yang–Baxter equation. There are only two independent solutions and one of them as expected is the same as $\hat{R}_\otimes$, and another denoted by $\hat{R}(x)$ is a new solution. This procedure is depicted in figure 2.

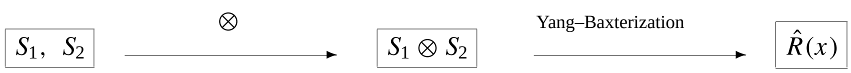

Figure 2. Tensor production and then Yang–Baxterization.

$\hat{R}_2$ is symmetric with the non-zero elements

$$
\begin{aligned}
&\hat{R}_{1\ 1} = \hat{R}_{6\ 6} = \hat{R}_{11\ 11} = \hat{R}_{16\ 16} = (xq_2^2 - 1)(x^2q_1^2 - 1)\\
&\hat{R}_{2\ 2} = \hat{R}_{12\ 12} = (x^2q_1^2 - 1)(q_2^2 - 1)x\\
&\hat{R}_{2\ 5} = \hat{R}_{12\ 15} = (x^2q_1^2 - 1)(x - 1)q_2\\
&\hat{R}_{3\ 3} = \hat{R}_{8\ 8} = (xq_2^2 - 1)(q_1^2 - 1)x^2\\
&\hat{R}_{3\ 9} = \hat{R}_{8\ 14} = (xq_2^2 - 1)(x^2 - 1)q_1\\
&\hat{R}_{4\ 4} = (q_2^2 - 1)(q_1^2 - 1)x^3\\
&\hat{R}_{4\ 7} = (x - 1)(q_1^2 - 1)x^2q_2\\
&\hat{R}_{4\ 10} = (x^2 - 1)(q_2^2 - 1)xq_1\\
&\hat{R}_{4\ 13} = \hat{R}_{7\ 10} = (x + 1)(x - 1)^2q_2q_1\\
&\hat{R}_{5\ 5} = \hat{R}_{15\ 15} = (x^2q_1^2 - 1)(q_2^2 - 1)\\
&\hat{R}_{7\ 7} = (q_2^2 - 1)(q_1^2 - 1)x^2\\
&\hat{R}_{7\ 13} = (x^2 - 1)(q_2^2 - 1)q_1\\
&\hat{R}_{9\ 9} = \hat{R}_{14\ 14} = (xq_2^2 - 1)(q_1^2 - 1)\\
&\hat{R}_{10\ 10} = (q_2^2 - 1)(q_1^2 - 1)x\\
&\hat{R}_{10\ 13} = (x - 1)(q_1^2 - 1)q_2\\
&\hat{R}_{13\ 13} = (q_2^2 - 1)(q_1^2 - 1)
\end{aligned}
$$

and four eigenvalues $(xq_1^2 - 1)(xq_2^2 - 1)(x^2 - q_2^2)$, $-(xq_1^2 - 1)(x - q_2^2)(x^2 - q_2^2)$, $(x - q_1^2)(x - q_2^2)(x^2 - q_2^2)$ and $-(xq_2^2 - 1)(x - q_1^2)(x - q_2^2)$.

4. Discussions

It should be stressed here that the operations of tensor-production and Yang–Baxterization are non-commutative. This fact is demonstrated in figure 3.

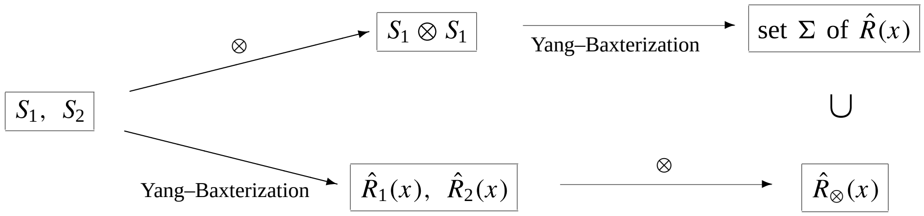

Figure 3. Non-commutativity of the two operations.

The set Σ is formed by the solutions obtained by parametrization of the tensored braid group representations S_1 and S_2, and it contains the solution $\hat{R}_1(x) \otimes \hat{R}_2(x)$ as a member. The latter is trivial and it states the fact that a tensor product of two solutions of the Yang–Baxter equation is still a solution of the Yang–Baxter equation. As Σ contains a trivial solution in any case, Σ must be non-empty.

In fact, the equivalence of $\hat{R}_\otimes(x)$ and $\hat{R}_1(x) \otimes \hat{R}_2(x)$, and the difference between $\hat{R}(x)$ and $\hat{R}_1(x) \otimes \hat{R}_2(x)$ can easily be verified and proved by checking their eigenvalues. The set of eigenvalues of $\hat{R}_\otimes(x)$ is a product set of the eigenvalue set of solutions $\hat{R}_1(x)$, $\hat{R}_2(x)$, while the eigenvalue set of solution $\hat{R}(x)$ has nothing to do with the eigenvalue set of $\hat{R}_1(x)$ and $\hat{R}_2(x)$.

By analysing the relationship between the eigenvalue sets, one can easily recognize whether the solutions generated by the knitting ansatz are trivial or not. Generically, if the ansatz generates more than one independent solutions, there must be non-trivial solution(s).

Similarly, by comparing the eigenvalues of $\hat{R}(x)$ and known 16×16 R-matrices, we learn that our R-matrix is new.

Finally, we emphasize that, though we applied the knitting ansatz to two braid group representations of the same dimension, it is not a requirement that the dimensions must be the same.

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